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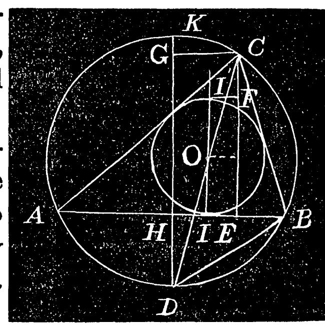
SOLUTIONS OF PROBLEMS IN NUMBER SIX, VOL. VII.

SOLUTIONS of problems in No. 6, Vol. VII, have been rec'd as follows: From Prof. W. P. Casey, 325, 326, 327, 329; Geo. M. Day, 325, 326, 328, 329; Prof. A. B. Evans, 327; Alexander Evans, 328; George Eastwood, 325, 326, 329; Wm. Hoover, 325, 326, 329; Prof. J. H. Kershner, 325, 326, 328; Prof. F. P. Matz, 325; Octavian L. Mathiot, 325; Prof. D. J. Mc Adam, 328; Prof. P. H. Philbrick, 325, 326, 328; Prof. E. B. Seitz, 325, 326, 328, 329, 330; Prof. J. Scheffer, 325, 326, 327, 328, 329.

325. "Given the altitude and radius of the circumscribed and inscribed circles of a plane scalene triangle; to find the three sides."

SOLUTION BY PROF. F. P. MATZ, KING'S MOUNTAIN, N. C.



Let ABC represent the triangle, CE, the given altitude, O the center of the inscribed circle, R and r radii of the circumscribed and insc'bd circles, respectively, and put $CE=a$.

Through O, draw CD, and draw DK perpendicular to AB; then is DK a diam. of the circumscribed circle parallel to CE. Also through O draw II' parallel to CE, and draw CG and $I'F$ each parallel to AB. Then (Eucl. Book VI., Prop. XXIV) we have

$$DH : IO(=r) :: OI'(=r) : FC(=a-2r);$$

$$\therefore DH = \frac{r^2}{a-2r}.$$

By property of the circle

$$AH=\sqrt{(DH\times HK)} = \frac{r\sqrt{[2R(a-2r)-r^2]}}{a-2r};$$

$$\therefore AB = \frac{2r\sqrt{[2R(a-2r)-r^2]}}{a-2r}.$$

Similarly we find

$$GC=HE=\frac{(a-r)\sqrt{[2R(a-2r)-(a-r)^2]}}{a-2r}.$$

Therefore we have

$$AC=[(AH+HE)^2+a^2]^{\frac{1}{2}}=\frac{(a-r)[2R(a-2r)-r^2]^{\frac{1}{2}}+r[2R(a-2r)-(a-r)^2]^{\frac{1}{2}}}{(a-2r)},$$

$$BC=[(AH-HE)^2+a^2]^{\frac{1}{2}}=\frac{(a-r)[2R(a-2r)-r^2]^{\frac{1}{2}}-r[2R(a-2r)-(a-r)^2]^{\frac{1}{2}}}{(a-2r)}.$$

326. "Draw a line bisecting a given triangle so that the part lying within shall be, 1st, a minimum, 2nd, a maximum."

SOLUTION BY GEO. M. DAY, LOCKPORT, N. Y.

Let a denote the area of the triangle, and x, y and z the sides of the triangle formed by the bisecting line; and put $\phi =$ the angle opposite z.
We have

$$xy\sin\phi = a; \tag{1}$$
$$z^2 = x^2+y^2-2xy\cos\phi. \tag{2}$$

Substituting in (2) the value of y from (1), and then differentiating and placing the result equal to zero, we get

$$x = \sqrt{\frac{a}{\sin\phi}} \text{ and from (1), } y = \sqrt{\frac{a}{\sin\phi}}.$$

Substituting these values for x and y in (2), and reducing we find

$$z^2 = 2a\tan\tfrac{1}{2}\phi.$$

Therefore for a maximum the bisecting line must be taken opposite the greatest angle, and for a minimum, opposite the least angle.

SOLUTION BY PROF. P. H. PHILBRICK, IOWA STATE UNIVERSITY.

Let ABC represent the triangle, A the least and C the greatest angle. Take $Am = x$, $An = y$; the sides as usual a, b, c; $mn = l$. Then is

$$l^2 = x^2+y^2-2xy\cos A. \tag{1}$$

This expression for given values of x and y is a minimum when A is least and a maximum when A is greatest or equal C.

Let mn represent the minimum line required and we also have

$$2xy = bc. \tag{2}$$

Substituting for x in (1), differentiating and equating to zero, we get

$$2y^2 = bc = 2xy, \therefore x = y = \sqrt{(\tfrac{1}{2}bc)}.$$

2. Let l_1 be the length of the maximum line, and v and w distances of its extremities from C, on CA and CB. Then

$$l_1^2 = v^2+w^2-2vw\cos C, \tag{4}$$
$$2vw = ab. \tag{5}$$

By putting $v = w + z$ and substituting we get $z^2+ab(1-\cos C)$ a max. And this occurs when $v = b$ and $w = \frac{1}{2}a$. A line from A to the middle of BC is therefore the line required.

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327. "The poles of the radical axis of two circles taken with respect to each circle, and the two centers of similitude of the circles, are four harmonic points. (Ex. 8, p. 367, Chauvanet's Modern Geom.)"

SOLUTION BY PROF A. B. EVANS, LOCKPORT, N. Y.

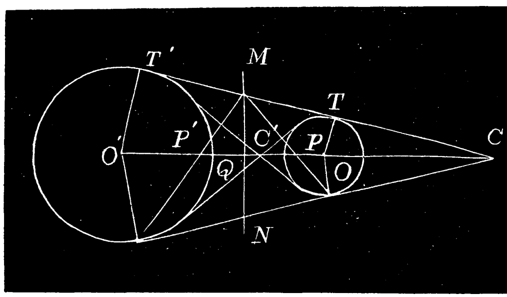
Let P and P' be the poles of the radical axis MN with respect to the circles O and O' whose centres of similitude are C and C'.

Put $OO' = a$, $O'T' = R$, $OT = r$; then

$$OC = \frac{ar}{R-r},\quad OC' = \frac{ar}{R+r},\quad O'C = \frac{aR}{R-r},\quad O'C' = \frac{aR}{R+r};$$

and since $O'Q^2—OQ^2 = R^2—r^2$ and $O'Q+OQ = a$,

$$OQ = \frac{a^2-R^2+r^2}{2a},\qquad O'Q = \frac{a^2+R^2-r^2}{2a}.$$



The radius being a mean proportional between the distances of the polar and its pole from the centre of the circle, $r^2 = OP \times OQ$, and $R^2 = O'P' \times O'Q$, and therefore $OP = 2ar^2 \div (a^2—R^2 + r^2)$, $O'P' = 2aR^2 \div (a^2+R^2—r^2)$.

Now $PC' = OC'—OP$ and $P'C' = O'C'—O'P'$; therefore

$$\frac{P'C'}{PC'} = \frac{R}{r}\left[\frac{a^2-R^2+r^2}{a^2+R^2-r^2}\right]. \qquad (1)$$

Again $PC = OC + OP$ and $P'C = O'C—O'P'$; therefore

$$\frac{P'C}{PC} = \frac{R}{r}\left[\frac{a^2-R^2+r^2}{a^2+R^2-r^2}\right]. \qquad (2)$$

From (1) and (2)

$$\frac{P'C'}{PC'} = \frac{P'C}{PC},$$

and therefore P, P' and C, C' are four harmonic points.

SOLUTION BY PROF. JOSEPH H. KERSHNER.

Let C and C be the centres of similitude, P and P' the poles, and R any point on the radical axis. Draw RP, RP' cutting the circles in A and A'. Connect AP', $A'P$ and RC', which will meet in a common point: See any good work on Mod. Geom. By the transversal CA' we have

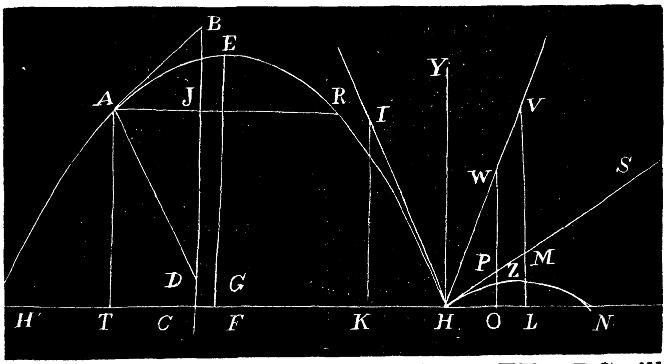
$$CP'.AP.A'R = CP.AR.A'P';$$

also $$C'P'.AP.A'R = C'P.AR.A'P'.$$

Dividing, $$CP' : C'P' :: CP : C'P.$$

328. "A body is projected from the top of a tower 100 ft high at an angle of elevation of 45°, with a velocity of 60 ft per second. Find the distance from the point at which it first strikes the horizontal plane to the second point at which it strikes the plane. The modulus of elasticity being $\frac{1}{3}$, and the resistance of the atmosphere neglected."

SOLUTION BY ALEXANDER EVANS, ESQ., ELKTON, MD.



Draw AR, a horizontal line through the top of the tower AT; make the angle $RAB = 45°$, and take $AB = 60$ equal the velocity at A. Cons't the parabola $AERH$, draw the principal axis EF, and through B, the line BC parallel to EF. BC will be the Hodograph to the parabola. [See *Elements of Dynamics*, by W. K. Clifford, F. R. S., p. 67.]

Draw the horizontal line THN through the bottom of the tower: and through H, the tangent HI; and draw AD parallel to HI to intersect the hodograph in D; AD will represent in magnitude and direction the velocity of the projectile at H.

Draw HY parallel to EF and make the angle YHW equal to YHI. From any point W draw WO parallel to EF, and take PO equal one-third of WO; draw from H the line $HPMS$, this will be the direction of the projectile after reflection.

Take HI on the tangent equal to AD, and draw IK parallel to EF. KH is the horizontal velocity at H; and since the horizontal velocity is not changed after the impact, take $HL = HK = AJ$; through L draw LV parallel to EF; VL will be the hodograph to HZN, and HM will represent in magnitude and direction the velocity of the projectile at H after rebounding.

With HM the velocity, and the angle MHN, construct the parabola HZN; HN will be the range on the rebound $= 79.7$ nearly.

As the velocity at the point H' on the other side of the principal axis, is the same as at H, if we suppose this velocity to be V, the time of flight in the first parabola will be $(2V \div g) \sin \angle IHK = (2V \div g) \sin \theta$; and in

the second parabola, $(2V \div g)e \sin\theta$, and as the velocities horizontally are uniform, the range in the second parabola will be to that in the first as e to 1, that is one-third of the first; and as the range in the first, $2GH =$ 239.2, that in the second $= 79.7$ nearly.

[Prof. Philbrick, and Prof. Scheffer, each gave a very elegant solution of this problem, and each shows that the range in the second parabola is to that in the first as e to 1; and Prof Scheffer draws the inferrence, that the body would rest, after an infinite number of rebounds, at a distance from $H' = [1 \div (1-e)] \times H'H$.]

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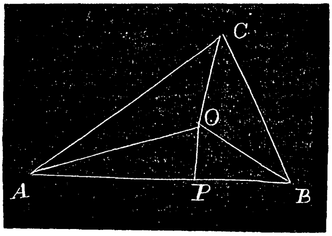
329. "Three points, A, B, C, being given, to find a point M, whose distance from A, B and C, shall be a minimum."

SOLUTION BY PROF. W. P. CASEY, SAN FRANCISCO, CAL.



Let ABC be the given triangle, a, b, c its sides, and O the required point. Put $AO = u$, $BO = v$, $CO = w$. Draw OP perpendicular to AB and let $AP = x$ and $PO = y$; also let $\angle AOP = \theta$, $BOP = \varphi$, $COP = \beta$.

Then $u^2 = x^2 + y^2$, $v^2 = (c-x)^2 + y^2$, and $w^2 = (b\cos A - x)^2 + (b\sin A - y)^2$.

And in order that $u+v+w$ may be a maximum we must have

$$\frac{du}{dx}+\frac{dv}{dx}+\frac{dw}{dx}=0,\quad \frac{du}{dy}+\frac{dv}{dy}+\frac{dw}{dy}=0.$$

Now $\dfrac{du}{dx} = \dfrac{x}{u} = \sin\theta$, $\dfrac{dv}{dx} = -\dfrac{c-x}{v} = -\sin\varphi$, $\dfrac{dw}{dx} = -\dfrac{b\cos A - x}{w}$ $= -\sin\beta$; $\dfrac{du}{dy} = \dfrac{y}{u} = \cos\theta$, $\dfrac{dv}{dy} = \dfrac{y}{v} = \cos\varphi$ and $\dfrac{dw}{dy} = -\dfrac{b\cos A - y}{w}$ $= \cos\beta$. Hence we have the conditions,

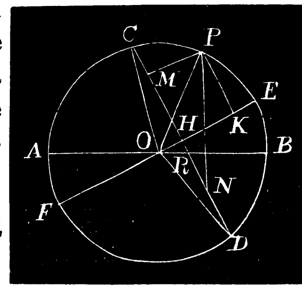
$$\sin\theta = \sin\varphi + \sin\beta,$$
$$\cos\theta = -\cos\varphi - \cos\beta.$$

Squaring and adding we find $\cos(\beta - \varphi) = -\frac{1}{2}$, or $\beta - \varphi = 120°$; $\therefore$ the angle $BOC = 120°$. And in a similar way it may be shown that $\angle AOC = 120° = \angle AOB$. Therefore if on any two sides of the triangle segments of a circle be described containing angles of 120° their intersect'n will determine the point O.

This problem is the same as Problem 257, ANALYST, Vol. VI, p. 93, if we take m, n, r in that problem each $= 1$. This problem possesses considerable interest in the history of mathematics.

330. "Two points are taken at random within a circle on opposite sides of a given diameter, and a third point is taken at random in the circumference; find the average area of the triangle formed by joining the points."

SOLUTION BY PROF. E. B. SEITZ, KIRKSVILLE, MO.



Let $ACBD$ be the given circle, AB the given diameter, M, N two random points on opposite sides of AB, CD the chord through them, P a random point in the circumference, and O the center of the circle. Draw the diameter EF perpendicular to CD, and PK perpendicu'r to EF.

Let $OA = r$, $RM = x$, $RN = y$, $RC = u$, $RD = v$, $\angle COH = \theta$, $BOH = \varphi$, and $POE = \psi$. Then we have

$$u = r\sin\theta + r\cos\theta\tan\varphi, \qquad v = r\sin\theta - r\cos\theta\tan\varphi,$$

$$\text{area } MNP = \tfrac{1}{2}r(x+y)(\cos\psi - \cos\theta) = u_1, \text{ when } \psi < \theta,$$

and $$\text{area } MNP = \tfrac{1}{2}r(x+y)(\cos\theta - \cos\psi) = u_2, \text{ when } \varphi > \theta,$$

An element of surface at M is $r\sin\theta\, d\theta dx$, and at N it is $(x+y)d\varphi dy$, and an element of the circumference at P is $rd\psi$. The limits of θ are 0 and $\frac{1}{2}\pi$; of φ, $-\theta$ and θ, and doubled; of x, 0 and u; of y, 0 and v; and of ψ, 0 and θ, and θ and π.

By limiting P to the semi-circumference ECF, the whole number of ways the three points can be taken is $\frac{1}{4}\pi^2 r^4.\pi r$; hence the required average is

$$\frac{8}{\pi^3 r^5}\int_0^{\frac{1}{2}\pi}\int_{-\theta}^{+\theta}\int_0^u\int_0^v\left\{\int_0^\theta u_1 r d\psi + \int_\theta^\pi u_2 r d\psi\right\} r\sin\theta\, d\theta d\varphi dx(x+y)dy$$

$$=\frac{4}{\pi^3 r^2}\int_0^{\frac{1}{2}\pi}\int_{-\theta}^{+\theta}\int_0^u\int_0^v(\pi - 2\theta + 2\tan\theta)\sin\theta\cos\theta d\theta d\varphi dx(x+y)^2 dy$$

$$=\frac{2r^2}{3\pi^3}\int_0^{\frac{1}{2}\pi}\int_{-\theta}^{+\theta}(\pi - 2\theta + 2\tan\theta)(7\tan^2\theta + \tan^2\varphi)(1 - \cos^2\theta\sec^2\varphi)\sin\theta\cos^3\theta d\theta d\varphi$$

$$=\frac{4r^2}{9\pi^3}\int_0^{\frac{1}{2}\pi}[(\pi-2\theta)\cos\theta + 2\sin\theta](24\theta\sin^2\theta - 3\theta + 3\sin\theta\cos\theta - 22\sin^3\theta\cos\theta)\sin\theta d\theta$$

$$=\frac{r^2}{\pi}\left[\frac{13}{18} + \frac{352}{81\pi^2}\right].$$

SOLUTION OF MISCEL. PROB. (2), P. 149, VOL. VII, BY PROF. SCHEFFER.

If in the series $S = ax + bx^2 + cx^3 + dx^4 + ex^5 + \ldots$ we put $x = y \div (1+vy)$, we get $S = ay(1+vy)^{-1} + by^2(1+vy)^{-2} + cy^3(1+vy^{-3} + dy^4(1+vy)^{-4} + \ldots$

Expanding by the Binomial Theorem, we obtain

$$ay - (av-b)y^2 + (av^2 - 2bv + c)y^3 - (av^3 - 3bv^2 + 3cv - d)y^4 + \ldots$$

Substituting for y its equiv't $x \div (1-vx)$, we have the req'd transformation.

[This poblem was solved in a similar manner by Prof. Kershner.]